The Hill determinant: an application to a class of confinement potentials

1983 J. Phys. A: Math. Gen. 16 209

(http://iopscience.iop.org/0305-4470/16/1/028)

J. Phys. A: Math. Gen. **16** (1983) 209–211. Printed in Great Britain

COMMENT

The Hill determinant: an application to a class of confinement potentials

R N Chaudhuri

Department of Physics, Visva-Bharati, Santiniketan 731235, West Bengal, India

Received 27 July 1982

Abstract. It is shown that the method of determination of the eigenvalues in terms of infinite continued fractions for a class of confinement potentials is equivalent to the vanishing of the Hill determinant. The problem of normalisation of the wavefunction is also discussed.

In a recent paper Datta and Mukherjee (1980) studied a class of confinement potentials of the form

$$V(r) = -a/r + br + cr^2 \qquad c > 0 \tag{1}$$

and obtained the radial Schrödinger equation in the form

$$g''(r) + f(r)g'(r) + \phi(r)g(r) = 0 \tag{2}$$

by making the standard transformation

$$R(r) = r^{l+1}\exp(-\tfrac{1}{2}\alpha r^2 - \beta r)g(r) \tag{3}$$

where $\alpha(>0)$ and β are constants and l is the relative angular momentum. It was shown by Flessas (1982) that the forms of $f(r)$ and $\phi(r)$ are such that $r = 0$ is a regular singular point and $r = \infty$ is an irregular singular point of the differential equation (2). The indicial equation has the roots 0 and $-(2l+1)$ of which the latter is discarded since it does not satisfy the proper boundary condition at $r = 0$. Therefore equation (2) admits a convergent series solution

$$g(r) = \sum_{n=0}^{\infty} p_n r^n \tag{4}$$

valid in the region $0 \leq r < \infty$. The coefficients p_n satisfy the difference equation

$$A_n p_{n+2} + B_n p_{n+1} + C_n p_n = 0 \tag{5}$$

with

$$p_{-1} = 0 \qquad \text{and} \qquad A_{-1}p_1 + B_{-1}p_0 = 0. \tag{6}$$

A_n, B_n and C_n are functions of n, l and the energy E. Equation (5) can be rewritten as

$$\frac{p_{n+1}}{p_n} = \frac{-C_n}{B_n + A_n p_{n+2}/p_{n+1}}. \tag{7}$$

0305-4470/83/010209+03$02.25

Repeated application of equation (7) for $n=0,1,2,\ldots,n-2$ gives the continued fraction

$$\frac{B_{-1}}{A_{-1}}=\cfrac{C_0}{B_0-\cfrac{A_0C_1}{\ddots\, B_{n-3}-\cfrac{A_{n-3}C_{n-2}}{B_{n-2}+A_{n-2}p_n/p_{n-1}}}}. \tag{8}$$

When $n\to\infty$ we get the infinite continued fraction, called the 'consistency condition' by Datta and Mukherjee (1980), for the existence of the solutions for the system of equations (5). The solutions of (8) as $n\to\infty$ will give the energy eigenvalues of the problem. It has been rightly pointed out by Flessas (1982) that not all the eigenvalues should be allowed since the boundary condition $R(r)\to 0$ as $r\to\infty$ is not incorporated into the method. So along with the 'consistency condition' the condition for the normalisability of the wavefunction should also be imposed. The statement made by Flessas (1982) that the correct eigenvalues would be obtained solely from the requirement that $g(r)$ as $r\to\infty$ does not compensate $\exp(-\frac{1}{2}\alpha r^2-\beta r)$ is, however, not correct. We consider the absolute series for $g(r)$. From equation (3) we find that $R(r)$ is normalisable when

$$g(r)\leqslant M\exp(\tfrac{1}{2}\alpha r^2+\beta r)/r^{l+2} \qquad r\to\infty \tag{9}$$

where M is a constant. By making the series expansion and comparing the coefficients of r^k we get the following bound on $|p_k|$:

$$|p_k|\leqslant\sum_{n=0}^{L}M(\tfrac{1}{2}\alpha)^n\frac{\beta^{k-2n+l+2}}{n!(k-2n+l+2)!} \tag{10}$$

where $L=\frac{1}{2}(k+l+2)$ or $\frac{1}{2}(k+l+1)$ whichever is an integer. Equation (10) gives us the condition for normalisation of the wavefunction.

Next we apply the method of the Hill determinant for the calculation of eigenvalues. The eigenvalues of anharmonic oscillators of type λx^{2m} have been calculated by Biswas *et al* (1971, 1973) using the infinite Hill determinant method. The method is non-perturbative and the eigenvalues can be calculated to a high degree of accuracy for any arbitrary value of the coupling constant λ. Recently Saxena and Varma (1982) studied the ground-state energy of the s-wave hydrogen atom with the polynomial perturbation series and showed that the results agree well with the variational and Hill determinant calculations. We shall show that the Hill determinant method gives equation (8) as $n\to\infty$ and therefore (8) is the correct eigenvalue equation in conjunction with the normalisation condition (10).

From (5) and (6) we obtain p_n in terms of p_0

$$p_n=(-1)^nD_np_0/(A_{-1}A_0A_1\ldots A_{n-2}) \tag{11}$$

where D_n is an $n\times n$ determinant:

$$D_n=\begin{vmatrix} B_{-1} & A_{-1} & 0 & 0 & \ldots & & & \\ C_0 & B_0 & A_0 & 0 & \ldots & & & \\ 0 & C_1 & B_1 & A_1 & \ldots & & & \\ \vdots & \vdots & \vdots & & \ddots & & & \\ & & & & 0 & C_{n-3} & B_{n-3} & A_{n-3} \\ & & & & 0 & 0 & C_{n-2} & B_{n-2} \end{vmatrix}. \tag{12}$$

D_n as $n \to \infty$ is the Hill determinant. The necessary and sufficient condition that non-trivial p_n exist which solve (5) is that the infinite Hill determinant vanishes. The zeros of D_n in the energy parameter will determine the eigenvalues of the problem when $n \to \infty$. The D_n satisfy the following difference equation

$$D_n = B_{n-2}D_{n-1} - C_{n-2}A_{n-3}D_{n-2}. \tag{13}$$

The connection between the determinant and the continued fraction is that when $D_n = 0$ the corresponding continued fraction (8) is truncated since $p_n = 0$. The vanishing of the Hill determinant corresponds to the equation

$$\frac{B_{-1}}{A_{-1}} = \begin{vmatrix} C_0 & A_0 & 0 & 0 & \dots \\ 0 & B_1 & A_1 & 0 & \dots \\ 0 & C_2 & B_2 & A_2 & \dots \\ . & . & . & . & \dots \end{vmatrix} \Bigg/ \begin{vmatrix} B_0 & A_0 & 0 & 0 & \dots \\ C_1 & B_1 & A_1 & 0 & \dots \\ 0 & C_2 & B_2 & A_2 & \dots \\ . & . & . & . & \dots \end{vmatrix}$$

$$= \cfrac{C_0}{B_0 - \cfrac{A_0 C_1}{\ddots}} . \tag{14}$$

So starting with the vanishing of the Hill determinant we arrive at the same infinite continued fraction as given in (8) in the limit $n \to \infty$. So we find that, in the limit $n \to \infty$ or the vanishing of the Hill determinant, equation (8) gives the correct eigenvalues when $R(r) \to 0$ as $r \to \infty$. In terms of the determinant the condition of normalisation of the wavefunction is

$$|D_k| \leqslant \sum_{n=0}^{L} M'(\tfrac{1}{2}\alpha)^n \beta^{k-2n+l+2} \frac{A_{-1}A_0 \dots A_{k-2}}{(n!(k-2n+l+2)!)} \tag{15}$$

where $M' = M/p_0$.

For each eigenvalue as determined by (8) or the vanishing of the Hill determinant (equation (12)) in the limit $n \to \infty$ the normalisation condition (10) or (15) should be checked. The equations are valid for all values of l.

References

Biswas S N, Datta K, Saxena R P, Srivastava P K and Varma V S 1971 *Phys. Rev.* D **4** 3617–20
—— 1973 *J. Math. Phys.* **14** 1190–5
Datta D P and Mukherjee S 1980 *J. Phys. A: Math. Gen.* **13** 3161–70
Flessas G P 1982 *J. Phys. A: Math. Gen.* **15** L1–5
Saxena R P and Varma V S 1982 *J. Phys. A: Math. Gen.* **15** L149–53